

Eigenvalues of a H -generalized join graph operation constrained by vertex subsets

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Abstract

Considering a graph H of order p , a generalized H -join operation of a family of graphs G_1, \dots, G_p , constrained by a family of vertex subsets $S_i \subseteq V(G_i)$, $i = 1, \dots, p$, is introduced. When each vertex subset S_i is (k_i, τ_i) -regular, it is deduced that all non-main adjacency eigenvalues of G_i , different from $k_i - \tau_i$, for $i = 1, \dots, p$, remain as eigenvalues of the graph G obtained by the above mentioned operation. Furthermore, if each graph G_i of the family is k_i -regular, for $i = 1, \dots, p$, and all the vertex subsets are such that $S_i = V(G_i)$, the H -generalized join operation constrained by these vertex subsets coincides with the H -generalized join operation. Some applications on the spread of graphs are presented. Namely, new lower and upper bounds are deduced and a infinity family of non regular graphs of order n with spread equals n is introduced.

AMS classification: 05C50, 15A18

Keywords: Graph eigenvalues, spread of a graph; adjacency matrix;

*Work supported by *FEDER* funds through *COMPETE*–Operational Programme Factors of Competitiveness and by Portuguese funds through the *Center for Research and Development in Mathematics and Applications* (University of Aveiro) and the Portuguese Foundation for Science and Technology (“FCT–Fundação para a Ciência e a Tecnologia”), within project PEst-C/MAT/UI4106/2011 with *COMPETE* number FCOMP-01-0124-FEDER-022690 and also to the project PTDC/MAT/112276/2009. These authors also thanks the hospitality of Departamento de Matemáticas, Universidad Católica del Norte, Chile, during the visit of which this research was finalized.

[†]Research partially supported by Fondecyt - IC Project 11090211, Chile.

[‡]Research supported by Project Fondecyt 1100072, Chile. [§]These authors thanks the hospitality of Departamento de Matemática, Universidade de Aveiro, Aveiro, Portugal, in which this research was started.

1 Notation and main concepts

We deal with undirected simple graphs herein simply called graphs. For each graph G , the vertex set is denoted by $V(G)$ and its edge set by $E(G)$. Usually, we consider that the graph G has order n , that is $V(G) = \{1, \dots, n\}$. An edge with end vertices i and j is denoted by ij and then we say that the vertices i and j are adjacent or neighbors. The number of neighbors of a vertex i is the degree of i and the neighborhood of i is the set of its neighbors, $N_G(i) = \{j \in V(G) : ij \in E(G)\}$. The maximum and minimum degree of the vertices of G is denoted by $\Delta(G)$ and $\delta(G)$, respectively. The complement of G , denoted by \overline{G} is such that $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{ij : ij \notin E(G)\}$. A path of length $p - 1$, P_p , in G is a sequence of vertices i_1, \dots, i_p all distinct except, eventually the first and the last) and such that $i_j i_{j+1} \in E(G)$, for $j = 1, \dots, p - 1$. If $i_1 = i_p$, then it is a closed path usually called cycle of length p and denoted C_p . A graph G is connected if there is a path between each pair of distinct vertices. A complete graph of order n , where each pair of distinct vertices is connected by an edge, is denoted by K_n . The complement of K_n , \overline{K}_n , is called the null graph. A graph G is bipartite if $V(G)$ can be partitioned into two subsets V_1 and V_2 such that every edge of G has one end vertex in V_1 and the other one in V_2 . This graph G is called complete bipartite and it is denoted $K_{p,q}$, if $|V_1| = p$, $|V_2| = q$ and each vertex of V_1 is connected with every vertex of V_2 .

The adjacency matrix of a graph G , $A(G) = (a_{i,j})$, is the $n \times n$ matrix

$$a_{i,j} = \begin{cases} 1 & \text{if } ij \in E(G) \\ 0 & \text{otherwise.} \end{cases}$$

Then the matrix $A(G)$ is a nonnegative symmetric with entries which are 0 and 1 and then all of its eigenvalues are real. Furthermore, since all its diagonal entries are equal to 0, the trace of $A(G)$ is zero. If G has at least one edge, then $A(G)$ has a negative eigenvalue not greater than -1 and a positive eigenvalue not less than the average degree of the vertices of G . Considering any matrix M we denote its spectrum (the multiset of the eigenvalues of M) by $\sigma(M)$. The spectrum of the adjacency matrix of a graph G , $\sigma(A(G))$, is simply denoted by $\sigma(G)$ and the eigenvalues of $A(G)$ are also called the eigenvalues of G . An eigenvalue λ of a graph G is called non-main if its associated eigenspace, denoted $\varepsilon_G(\lambda)$, is orthogonal to the all one vector, otherwise is called main.

Usually, the multiplicities of the eigenvalues are represented in the multiset $\sigma(G)$ as powers in square brackets. For instance, $\sigma(G) = \{\lambda_1^{[m_1]}, \dots, \lambda_q^{[m_q]}\}$ denotes that λ_1 has multiplicity m_1 , λ_2 has multiplicity m_2 , and so on. Throughout the paper, the eigenvalues of a graph G with n vertices, $\lambda_1(G), \dots, \lambda_n(G)$, are ordered as follows: $\lambda_1(G) \geq \dots \geq \lambda_n(G)$. If λ is an eigenvalue of the graph G and u is an associated eigenvector, the pair (λ, u) is called an eigenpair of G .

Considering a graph G of order n and a vertex subset $S \subseteq V(G)$, the characteristic vector of S is the vector $x_S \in \{0, 1\}^n$ such that $(x_S)_v = \begin{cases} 1 & \text{if } v \in S \\ 0 & \text{otherwise.} \end{cases}$

A vertex subset S is (k, τ) -regular if S induces a k -regular graph in G and every

vertex out of S has τ neighbors in S , that is,

$$|N_G(i) \cap S| = \begin{cases} k & \text{if } i \in S \\ \tau & \text{otherwise.} \end{cases}$$

When the graph G is k -regular, for convenience, $S = V(G)$ is considered $(k, 0)$ -regular. There are several properties of graphs related with (k, τ) -regular sets (see [2, 3]). For instance, we may refer the following properties:

- A graph G has a perfect matching if and only if its line graph has a $(0, 2)$ -regular set.
- A graph G is Hamiltonian if and only if its line graph has a $(2, 4)$ -regular set inducing a connected graph.
- A graph G of order n is strongly regular with parameters (n, p, a, b) if and only if $\forall v \in V(G)$ the vertex subset $S = N_G(v)$ is (a, b) -regular in $G - v$ (where $G - v$ is the graph obtained from G deleting the vertex v).

2 Generalized join graph operation with vertex subset constraints

Considering two vertex disjoint graphs G_1 and G_2 , the join of G_1 and G_2 is the graph $G_1 \vee G_2$ such that $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup \{xy : x \in V(G_1) \wedge y \in V(G_2)\}$. A generalization of the join operation was first introduced in [10] under the designation of *generalized composition* and more recently in [1] with the designation of *H-join*, defined as follows:

Consider a family of p graphs, $\mathcal{F} = \{G_1, \dots, G_p\}$, where each graph G_j has order n_j , for $j = 1, \dots, p$, and a graph H such that $V(H) = \{1, \dots, p\}$. Each vertex $j \in V(H)$ is assigned to the graph $G_j \in \mathcal{F}$. The H -join (generalized composition) of G_1, \dots, G_p is the graph $G = \bigvee_H \{G_j : j \in V(H)\}$ ($H[G_1, \dots, G_p]$) such that $V(G) = \bigcup_{j=1}^p V(G_j)$ and

$$E(G) = \left(\bigcup_{j=1}^p E(G_j) \right) \cup \left(\bigcup_{rs \in E(H)} \{uv : u \in V(G_r), v \in V(G_s)\} \right).$$

Now, we generalize the above H -join operation according to the next definition.

Definition 1 Consider a graph H of order p and a family of p graphs $\mathcal{F} = \{G_1, \dots, G_p\}$. Consider also a family of vertex subsets $\mathcal{S} = \{S_1, \dots, S_p\}$, such that $S_i \subseteq V(G_i)$ for $i = 1, \dots, p$. The H -generalized join operation of the family of graphs \mathcal{F} constrained by the family of vertex subsets \mathcal{S} , denoted by $\bigvee_{(H, \mathcal{S})} \mathcal{F}$,

produces a graph such that

$$\begin{aligned} V\left(\bigvee_{(H,\mathcal{S})} \mathcal{F}\right) &= \bigcup_{i=1}^p V(G_i), \\ E\left(\bigvee_{(H,\mathcal{S})} \mathcal{F}\right) &= \left(\bigcup_{i=1}^p E(G_i)\right) \cup \{xy : x \in S_i, y \in S_j, ij \in E(H)\}. \end{aligned}$$

Notice that the particular case of the H -generalized join operation of the family of graphs $\mathcal{F} = \{G_1, \dots, G_p\}$ constrained by the family of vertex subsets $\mathcal{S} = \{V(G_1), \dots, V(G_p)\}$, coincides with the above described H -generalized join operation.

Example 1 The Figure 1 depicts an example of a H -generalized join operation, with $H = P_3$, of a family of graphs $\mathcal{F} = \{G_1, G_2, G_3\}$ constrained by the family of vertex subsets $\mathcal{S} = \{S_1, S_2, S_3\}$, where $S_1 = \{a, b\}$, $S_2 = \{d, f\}$, and $S_3 = \{g, i, j\}$.

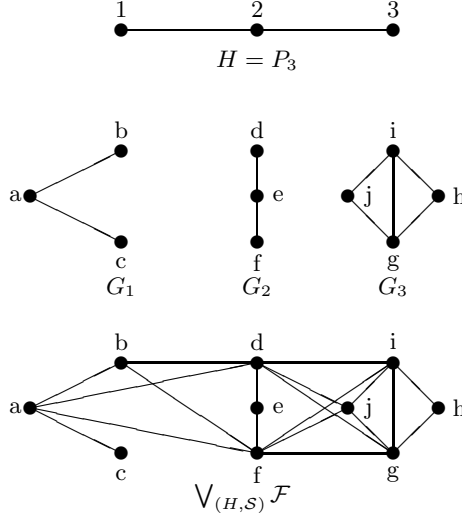


Figure 1: The H -generalized join operation of the family of graphs $\mathcal{F} = \{G_1, G_2, G_3\}$, constrained by the family of vertex subsets $\mathcal{S} = \{S_1, S_2, S_3\}$, where $S_1 = \{a, b\} \subset V(G_1)$, $S_2 = \{d, f\} \subset V(G_2)$ and $S_3 = \{g, i, j\} \subset V(G_3)$.

Now it is worth to recall the following result.

Lemma 1 [2] Let G be a graph with a (κ, τ) -regular set S , where $\tau > 0$, and $\lambda \in \sigma(A(G))$. Then, denoting the characteristic vector of S by \mathbf{x}_S , λ is non-main if and only if

$$\lambda = \kappa - \tau \quad \text{or} \quad \mathbf{x}_S \in (\mathcal{E}_G(\lambda))^\perp,$$

where $(\mathcal{E}_G(\lambda))^\perp$ denotes the vector space orthogonal to the eigenspace associated to the eigenvalue λ .

From now on, given a graph H , we denote

$$\delta_{i,j}(H) = \begin{cases} 1 & \text{if } ij \in E(H) \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 1 Consider a graph H of order p and a family of p graphs $\mathcal{F} = \{G_1, \dots, G_p\}$ such that $|V(G_i)| = n_i, i = 1, \dots, p$. Consider also the family of vertex subsets $\mathcal{S} = \{S_1, \dots, S_p\}$, where

$S_i \in \{S'_i \subseteq V(G_i) : \text{either } S'_i \text{ or } V(G_i) \setminus S'_i \text{ is } (k_i, \tau_i)\text{-regular for some integers } k_i, \tau_i\}$,

for $i = 1, \dots, p$. Let $G = \bigvee_{(H, \mathcal{S})} \mathcal{F}$. If $\lambda \in \sigma(G_i) \setminus \{k_i - \tau_i\}$ for some $i \in \{1, \dots, p\}$ is non-main, then $\lambda \in \sigma(G)$.

Proof. Denoting $\delta_{i,j} = \delta_{i,j}(H)$, then $\delta_{ij} x_{S_i} x_{S_j}^T$, where x_{S_i} and x_{S_j} are the characteristic vectors of S_i and S_j , respectively, is an $n_i \times n_j$ matrix whose entries are zero if $ij \notin E(H)$, otherwise

$$\left(\delta_{i,j} x_{S_i} x_{S_j}^T \right)_{q,r} = \begin{cases} 1 & \text{if } q \in S_i \wedge r \in S_j \\ 0 & \text{otherwise.} \end{cases},$$

Then the adjacency matrix of G has the form

$$A(G) = \begin{pmatrix} A(G_1) & \delta_{1,2} x_{S_1} x_{S_2}^T & \cdots & \delta_{1,p-1} x_{S_1} x_{S_{p-1}}^T & \delta_{1,p} x_{S_1} x_{S_p}^T \\ \delta_{2,1} x_{S_2} x_{S_1}^T & A(G_2) & \cdots & \delta_{2,p-1} x_{S_2} x_{S_{p-1}}^T & \delta_{2,p} x_{S_2} x_{S_p}^T \\ \delta_{3,1} x_{S_3} x_{S_1}^T & \delta_{3,2} x_{S_3} x_{S_2}^T & \cdots & \delta_{3,p-1} x_{S_3} x_{S_{p-1}}^T & \delta_{3,p} x_{S_3} x_{S_p}^T \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \delta_{p-1,1} x_{S_{p-1}} x_{S_1}^T & \delta_{p-1,2} x_{S_{p-1}} x_{S_2}^T & \cdots & A(G_{p-1}) & \delta_{p-1,p} x_{S_{p-1}} x_{S_p}^T \\ \delta_{p,1} x_{S_p} x_{S_1}^T & \delta_{p,2} x_{S_p} x_{S_2}^T & \cdots & \delta_{p,p-1} x_{S_p} x_{S_{p-1}}^T & A(G_p) \end{pmatrix}.$$

Let u_i be an eigenvector of $A(G_i)$ associated to the non-main eigenvalue $\lambda_i \neq k_i - \tau_i$, with $1 \leq i \leq p$. Then,

$$A(G) \begin{pmatrix} 0 \\ \vdots \\ 0 \\ u_i \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \delta_{1,i} (x_{S_i}^T u_i) x_{S_1} \\ \vdots \\ \delta_{i-1,i} (x_{S_i}^T u_i) x_{S_{i-1}} \\ A(G_i) u_i \\ \delta_{i+1,i} (x_{S_i}^T u_i) x_{S_{i+1}} \\ \vdots \\ \delta_{p,i} (x_{S_i}^T u_i) x_{S_p} \end{pmatrix} = \lambda_i \begin{pmatrix} 0 \\ \vdots \\ 0 \\ u_i \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad (1)$$

since x_{S_i} is the characteristic vector of the vertex subset S_i and S_i or $V(G_i) \setminus S_i$ is (k_i, τ_i) -regular (take into account that λ_i is non-main and then we may apply Lemma 1). ■

From the proof of Theorem 1, we may conclude the following corollary.

Corollary 1 Consider a graph H of order p and a family of p graphs $\mathcal{F} = \{G_1, \dots, G_p\}$ such that $|V(G_i)| = n_i, i = 1, \dots, p$. Consider also the family of vertex subsets $\mathcal{S} = \{V(G_1), \dots, V(G_p)\}$. Let $G = \bigvee_{(H, \mathcal{S})} \mathcal{F}$. If $\lambda \in \sigma(G_i)$ for some $i \in \{1, \dots, p\}$ is non-main, then $\lambda \in \sigma(G)$.

Proof. Consider an eigenpair (λ, u) of a graph G_i , for some $i \in \{1, \dots, p\}$, where λ is non-main. Then, taking into account the equations (1) where, in this case, x_{S_i} is the all one vector, the result follows. ■

Notice that in the above corollary, $G = \bigvee_{(H, \mathcal{S})} \mathcal{F}$ coincides with the H -join operation of the family of graphs \mathcal{F} [1] (generalized composition $H[G_1, \dots, G_p]$ in the terminology of [10]).

Example 2 Consider the Example 1, where $V(G_1) = \{a, b, c\}$ and $S_1 = \{a, b\}$ is $(1, 1)$ -regular, $V(G_2) = \{d, e, f\}$ and $S_2 = \{d, f\}$ is $(0, 2)$ -regular, $V(G_3) = \{g, h, i, j\}$ and $S_3 = \{g, i, j\}$ is $(2, 2)$ -regular.

- The eigenpairs of $A(G_1)$ are $\left(\sqrt{2}, \begin{bmatrix} \sqrt{2} \\ 1 \\ 1 \end{bmatrix}\right), \left(0, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right)$ and $\left(-\sqrt{2}, \begin{bmatrix} -\sqrt{2} \\ 1 \\ 1 \end{bmatrix}\right)$.
- The eigenpairs of $A(G_2)$ are $\left(\sqrt{2}, \begin{bmatrix} \sqrt{2} \\ 2 \\ \sqrt{2} \end{bmatrix}\right), \left(0, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right)$ and $\left(-\sqrt{2}, \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}\right)$.
- The eigenpairs of $A(G_3)$ are $\left(\frac{1+\sqrt{17}}{2}, \begin{bmatrix} \frac{1+\sqrt{17}}{4} \\ 1 \\ \frac{1+\sqrt{17}}{4} \end{bmatrix}\right), \left(0, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right), \left(\frac{1-\sqrt{17}}{2}, \begin{bmatrix} \frac{1-\sqrt{17}}{4} \\ 1 \\ \frac{1-\sqrt{17}}{4} \end{bmatrix}\right)$
and $\left(-1, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}\right)$.

Let $G = \bigvee_{(H, \mathcal{S})} \mathcal{F}$, where $H = P_3$. Then, denoting $\delta_{i,j} = \delta_{i,j}(H)$ and defining the characteristic vectors of the vertex subsets S_1, S_2 and S_3 considering their elements by alphabetic order, we obtain:

$$\begin{aligned} A(G) &= \begin{pmatrix} A(G_1) & \delta_{1,2}x_{S_1}x_{S_2}^T & \delta_{1,3}x_{S_1}x_{S_3}^T \\ \delta_{2,1}x_{S_2}x_{S_1}^T & A(G_2) & \delta_{2,3}x_{S_2}x_{S_3}^T \\ \delta_{3,1}x_{S_3}x_{S_1}^T & \delta_{3,2}x_{S_3}x_{S_2}^T & A(G_3) \end{pmatrix} \\ &= \begin{pmatrix} A(G_1) & \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} [1 \ 0 \ 1] & \mathbf{0}_{3 \times 4} \\ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} [1 \ 1 \ 0] & A(G_2) & \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} [1 \ 0 \ 1 \ 1] \\ \mathbf{0}_{4 \times 3} & \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} [1 \ 0 \ 1] & A(G_3) \end{pmatrix}. \end{aligned}$$

According to Theorem 1, $\{0, -1\} \subset \sigma(A(G))$. Notice that $S_1 \subseteq V(G_1)$ is $(1, 1)$ -regular and thus we are not able to get a conclusion about if the eigenvalue 0 of $A(G_1)$ is or not an eigenvalue of $A(G)$. On the other hand $S_2 \subseteq V(G_2)$ is $(0, 2)$ -regular and $S_3 \subseteq V(G_3)$ is $(2, 2)$ -regular. In fact,

$$\sigma(A(G)) = \{4.44999, 1.86239, 0, 0, 0, -1, -1.3822, -1.51442, -3.02546\}.$$

Consider a graph H of order p , a family of graphs $\mathcal{F} = \{G_1, \dots, G_p\}$, where each graph G_i has order n_i , and a family of vertex subsets $\mathcal{S} = \{S_1, \dots, S_p\}$, where for each $i \in \{1, \dots, p\}$, $S_i \subseteq V(G_i)$. If $G = \bigvee_{(H, \mathcal{S})} \mathcal{F}$ and (λ, \hat{u}) is an

eigenpair of $A(G)$, decomposing \hat{u} such that $\hat{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_p \end{pmatrix}$, where each u_i is a subvector of \hat{u} with n_i components, then $\lambda \hat{u} = A(G) \hat{u}$, that is,

$$\lambda \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_p \end{pmatrix} = \begin{pmatrix} A(G_1)u_1 + \left(\sum_{j \neq 1} \delta_{1,j} x_{S_j}^T u_j \right) x_{S_1} \\ A(G_2)u_2 + \left(\sum_{j \neq 2} \delta_{2,j} x_{S_j}^T u_j \right) x_{S_2} \\ \vdots \\ A(G_p)u_p + \left(\sum_{j \neq p} \delta_{p,j} x_{S_j}^T u_j \right) x_{S_p} \end{pmatrix}, \quad (2)$$

where $\delta_{i,j} = \delta_{i,j}(H)$.

Furthermore, if we assume that G_i is d_i -regular and S_i or its complement is (k_i, τ_i) -regular, for $i = 1, \dots, p$, respectively, according to Theorem 1,

$$\bigcup_{i=1}^p (\sigma(G_i) \setminus \{d_i, k_i - \tau_i\}) \subset \sigma(G),$$

since by one hand, as it is well known, all the eigenvalues of each graph G_i are non-main but d_i , on the other hand, if a regular graph has a (k, τ) -regular vertex subset, then $k - \tau$ is a non-main eigenvalue [2].

Additionally, assuming that $S_i = V(G_i)$, for $i = 1, \dots, p$, the remaining eigenvalues of G can be computed as follows: let us define \hat{u} , setting each of its subvectors $u_i = \theta_i e_{n_i}$, for $i = 1, \dots, p$, where each e_{n_i} is an all one vector with n_i componentes and $\theta_1, \dots, \theta_p$ are scalars. Then the system (2) becomes

$$\lambda \begin{pmatrix} \theta_1 e_{n_1} \\ \theta_2 e_{n_2} \\ \vdots \\ \theta_p e_{n_p} \end{pmatrix} = \begin{pmatrix} \left(d_1 \theta_1 + \sum_{j \neq 1} \delta_{1,j} \theta_j n_j \right) e_{n_1} \\ \left(d_2 \theta_2 + \sum_{j \neq 2} \delta_{2,j} \theta_j n_j \right) e_{n_2} \\ \vdots \\ \left(d_p \theta_p + \sum_{j \neq p} \delta_{p,j} \theta_j n_j \right) e_{n_p} \end{pmatrix}.$$

Therefore, (λ, \hat{u}) is an eigenpair for $A(G)$ if and only if $(\lambda, \hat{\theta})$, where $\hat{\theta} = (\theta_1, \theta_2, \dots, \theta_p)^T$, is an eigenpair of the matrix

$$M = \begin{pmatrix} d_1 & \delta_{1,2}n_2 & \dots & \delta_{1,p}n_p \\ \delta_{2,1}n_1 & d_2 & \dots & \delta_{2,p}n_p \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{p,1}n_1 & \delta_{p,2}n_2 & \dots & d_p \end{pmatrix}, \quad (3)$$

that is, $M\hat{\theta} = \lambda\hat{\theta}$.

Setting $D = \text{diag}(d_1, \dots, d_p)$ and $N = \text{diag}(n_1, \dots, n_p)$, then $M = A(H)N + D$ is similar to the symmetric matrix

$$M' = \begin{pmatrix} d_1 & \delta_{1,2}\sqrt{n_1n_2} & \dots & \delta_{1,p}\sqrt{n_1n_p} \\ \delta_{2,1}\sqrt{n_1n_2} & d_2 & \dots & \delta_{2,p}\sqrt{n_2n_p} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{p,1}\sqrt{n_1n_p} & \delta_{p,2}\sqrt{n_2n_p} & \dots & d_p \end{pmatrix}, \quad (4)$$

since $M' = KMK^{-1}$ with $K = \text{diag}(\sqrt{n_1}, \dots, \sqrt{n_p})$. Therefore, $M' = D + KA(H)K$ and $\sigma(M) = \sigma(M')$.

Based on the above analysis, we are able to deduce the following result.

Theorem 2 *Consider a graph H of order p and a family of regular graphs $\mathcal{F} = \{G_1, \dots, G_p\}$, where each regular graph G_i has degree d_i and order n_i . Consider the family of vertex subsets $\mathcal{S} = \{S_1, \dots, S_p\}$, where*

$$S_i \in \{S'_i \subseteq V(G_i) : S'_i \text{ or } V(G_i) \setminus S'_i \text{ is } (k_i, \tau_i)\text{-regular, for some } k_i, \tau_i\},$$

for $i = 1, \dots, p$. Assume that $G = \bigvee_{(H, \mathcal{S})} \mathcal{F}$ and M' is the matrix defined in (4). If $S_i = V(G_i)$, for $i = 1, \dots, p$, then

$$\sigma(G) = \left(\bigcup_{i=1}^p \sigma(G_i) \setminus \{d_i\} \right) \cup \sigma(M'),$$

otherwise $\sigma(G) \supseteq \bigcup_{i=1}^p \sigma(G_i) \setminus \{d_i, k_i - \tau_i\}$.

Proof. The conclusions are direct consequence of the above analysis, taking into account that if $S_i = V(G_i)$, for $i = 1, \dots, p$, then each S_i is (k_i, τ_i) -regular, with $k_i = d_i$ and $\tau_i = 0$. ■

3 Some applications on the spread of graphs

3.1 Definitions and basic results

Given a $n \times n$ complex matrix M , the spread of M , $s(M)$, is defined as $\max_{i,j} |\lambda_i(M) - \lambda_j(M)|$, where the maximum is taken over all pairs of eigenvalues of M . Then

$$s(M) = \max_{x,y} (x^* M x - y^* M y) = \max_{i,j} m_{i,j} (\bar{x}_i x_j - \bar{y}_i y_j),$$

where z^* is the conjugate transpose of z and the maximum is taken over all pairs of unit vectors in \mathbb{C}^n .

Theorem 3 [8] $s(M) \leq \left(2 \sum_{i,j} |m_{i,j}|^2 - \frac{2}{n} |\sum_i m_{i,i}|^2\right)^{1/2}$, with equality if and only if M is a normal matrix (that is, such that $M^*M = MM^*$), with $n - 2$ of its eigenvalues all equal to the average of the remaining two.

Several results on the spread of normal and Hermitian matrices were presented in [6, 9].

In this paper, we consider only the spread of adjacency matrices of simple graphs and we define the spread of a graph G as the spread $s(A(G))$, which is simply denoted by $s(G)$. Therefore,

$$s(G) = \max_{i,j} \{|\lambda_i(G) - \lambda_j(G)|\},$$

where the maximum is taken over all pairs of eigenvalues of the adjacency matrix of G . If the graph G has order n , then $s(G) = \lambda_1(G) - \lambda_n(G)$ and replacing the matrix M of Theorem 3 by $A(G)$, it follows that

$$s(G) = \lambda_1(G) - \lambda_n(G) \leq \sqrt{4|E(G)|}. \quad (5)$$

Denoting the average degree of the vertices of G by $\bar{d}(G)$, from (5) it follows that

$$s(G) \leq \sqrt{2n\bar{d}(G)} < \sqrt{2n(n-1)} \quad (6)$$

if $n > 2$, since $\bar{d}(G) \leq n - 1$ and $\bar{d}(G) = n - 1$ if and only if $G = K_n$. Notice that $\sigma(K_n) = \{n - 1, (-1)^{[n-1]}\}$ and then $s(K_n) = n$. Furthermore,

$$\bar{d}(G) \leq \frac{n}{2} \Rightarrow s(G) \leq n. \quad (7)$$

In [4] the following upper bounds on the spread of a graph were obtained.

Theorem 4 [4] If G is a graph of order n , then

$$s(G) \leq \lambda_1(G) + \sqrt{2|E(G)| - \lambda_1^2(G)} \leq 2\sqrt{|E(G)|}. \quad (8)$$

Equality holds throughout if and only if equality holds in the first inequality; equivalently, if and only if $|E(G)| = 0$ or $G = K_{p,q}$, for some p and q .

Theorem 5 [4] If G is a regular graph of order n , then $s(G) \leq n$. Equality holds if and only if the complement of G , \bar{G} , is disconnected.

Additional results on the spread of graphs can be found in [4, 7].

3.2 The spread of the join of two graphs

Now it is worth to recall the join of two vertex disjoint graphs G_1 and G_2 which is the graph $G_1 \vee G_2$ obtained from their union connecting each vertex of G_1 to each vertex of G_2 . Considering this graph operation, as direct consequence of Theorem 2, we have the following corollaries. Notice that Corollary 2 is well known (see, for instance, [10]).

Corollary 2 *If G_i is a d_i -regular graph of order n_i , for $i = 1, 2$, then*

$$\sigma(G_1 \vee G_2) = \bigcup_{i=1}^2 (\sigma(A(G_i)) \setminus \{d_i\}) \cup \{\beta_1, \beta_2\}.$$

where β_1 and β_2 are eigenvalues of the matrix $M' = \begin{pmatrix} d_1 & \sqrt{n_1 n_2} \\ \sqrt{n_1 n_2} & d_2 \end{pmatrix}$, that is,

$$\beta_1 = \frac{d_1 + d_2 + \sqrt{(d_1 - d_2)^2 + 4n_1 n_2}}{2} \quad (9)$$

$$\beta_2 = \frac{d_1 + d_2 - \sqrt{(d_1 - d_2)^2 + 4n_1 n_2}}{2}. \quad (10)$$

Corollary 3 *Consider a d_i -regular graph of order n_i , for $i = 1, 2$, and the graph $G = G_1 \vee G_2$ of order $n = n_1 + n_2$. Then*

$$s(G) = \begin{cases} \sqrt{(d_1 - d_2)^2 + 4n_1 n_2}, & \text{if } \lambda_n(G) = \beta_2 \\ \frac{d_2 - d_1 + \sqrt{(d_1 - d_2)^2 + 4n_1 n_2}}{2} + s(G_1), & \text{if } \lambda_n(G) = \lambda_{n_1}(G_1) \\ \frac{d_1 - d_2 + \sqrt{(d_1 - d_2)^2 + 4n_1 n_2}}{2} + s(G_2), & \text{if } \lambda_n(G) = \lambda_{n_2}(G_2). \end{cases} \quad (11)$$

Furthermore, setting $R = \sqrt{(d_1 - d_2)^2 + 4n_1 n_2}$,

$$s(G) = \max\left\{R, \frac{d_2 - d_1 + R}{2} + s(G_1), \frac{d_1 - d_2 + R}{2} + s(G_2)\right\}.$$

Proof. According to Corollary 2, $\sigma(A(G)) = \bigcup_{i=1}^2 (\sigma(A(G_i)) \setminus \{d_i\}) \cup \{\beta_1, \beta_2\}$, where β_1 and β_2 have the values (9) and (10), respectively. On the other hand, $\lambda_{n_i}(G_i) = d_i - s(G_i)$, for $i = 1, 2$. Therefore, the equalities in (11) follows, as well as the second part. ■

Corollary 4 *Let G_i be a d_i -regular graph of order n_i , for $i = 1, 2$, and $G = G_1 \vee G_2$. If $|d_1 - d_2| > |n_1 - n_2|$, then $s(G) > n = n_1 + n_2$.*

Proof. By construction, it is immediate that the order of G is $n = n_1 + n_2$. Taking into account that β_1 and β_2 , in (9) and (10) of Corollary 2, respectively, are eigenvalues of the adjacency matrix of $G = G_1 \vee G_2$, then

$$s(G) \geq \beta_1 - \beta_2 = \sqrt{(d_1 - d_2)^2 + 4n_1 n_2} > n_1 + n_2 = n.$$

Notice that $\sqrt{(d_1 - d_2)^2 + 4n_1n_2} > n_1 + n_2 \Leftrightarrow (d_1 - d_2)^2 + 4n_1n_2 > n_1^2 + n_2^2 + 2n_1n_2 \Leftrightarrow (d_1 - d_2)^2 > (n_1 - n_2)^2$. ■

Considering the complete graph K_k , for which $\sigma(K_k) = \{(-1)^{[k-1]}, k-1\}$, and the null graph \overline{K}_{n-k} , for which $\sigma(\overline{K}_{n-k}) = \{0^{[n-k]}\}$, and denote the join of these graphs by $G(n, k)$ (that is $G(n, k) = K_k \vee \overline{K}_{n-k}$), according to Corollary 2, $\sigma(G(n, k)) = \{(-1)^{[k-1]}, 0^{[n-k-1]}, \beta_1, \beta_2\}$, with

$$\begin{aligned}\beta_1 &= \frac{k-1 + \sqrt{(k-1)^2 + 4k(n-k)}}{2} \\ \beta_2 &= \frac{k-1 - \sqrt{(k-1)^2 + 4k(n-k)}}{2}.\end{aligned}$$

Therefore, $s(G(n, k)) = \beta_1 - \beta_2 = \sqrt{(k-1)^2 + 4k(n-k)}$. Furthermore, when $\frac{n+1}{3} < k < n-1$, the hypothesis of Corollary 4 hold for these graphs and then $s(G(n, k)) > n$.

Theorem 6 [4] *Among the family of graphs $G(n, k) = K_k \vee \overline{K}_{n-k}$, with $1 \leq k \leq n-1$, the maximum of $s(G(n, k))$ is attained when $k = \lfloor 2n/3 \rfloor$.*

In [4] the following conjecture was checked by computer for graphs of order $n \leq 9$.

Conjecture 1 [4] *The maximum spread $s(n)$ of the graphs of order n is attained only by $G(n, \lfloor 2n/3 \rfloor)$, that is, $s(n) = \lfloor (4/3)(n^2 - n + 1) \rfloor^{1/2}$ and so $\frac{1}{\sqrt{3}}(2n-1) < s(n) < \frac{1}{\sqrt{3}}(2n-1) + \frac{\sqrt{3}}{4n-2}$.*

3.3 The spread of the generalized join of graphs

Throughout this subsection we consider a graph H of order p and a family of regular graphs $\mathcal{F} = \{G_1, \dots, G_p\}$, where each regular graph G_i has degree d_i and order n_i . We consider also $M = A(H)N + D$, where $N = \text{diag}(n_1, \dots, n_p)$ and $D = \text{diag}(d_1, \dots, d_p)$, and we define $d_{i^*} - s(G_{i^*}) = \min\{d_i - s(G_i) : i = 1, \dots, p\}$ and the matrix

$$P = \begin{pmatrix} 0 & \sqrt{n_1n_2} & \dots & \sqrt{n_1n_p} \\ \sqrt{n_1n_2} & 0 & \dots & \sqrt{n_2n_p} \\ \vdots & \vdots & \ddots & \vdots \\ \sqrt{n_1n_p} & \sqrt{n_2n_p} & \dots & 0 \end{pmatrix}.$$

Using the above notation, with the following theorems, we state upper and lower bounds on the spread of $G = \bigvee_H \mathcal{F}$.

Theorem 7 *If $G = \bigvee_H \mathcal{F}$, then*

$$s(G) = s(M) + \max_{1 \leq i \leq p} \{\lambda_p(M) + s(G_i) - d_i, 0\}. \quad (12)$$

Furthermore,

$$s(G) \geq n_{\downarrow} \left(s(H) - (\tilde{d}_{\uparrow} - \tilde{d}_{\downarrow}) \right) - (n_{\uparrow} - n_{\downarrow}) \left(\lambda_p(H) + \tilde{d}_{\uparrow} \right) \quad (13)$$

where $\tilde{d}_{\uparrow} = \max_{1 \leq i \leq p} \frac{d_i}{n_i}$ ($\tilde{d}_{\downarrow} = \min_{1 \leq i \leq p} \frac{d_i}{n_i}$), and $n_{\uparrow} = \max_{1 \leq i \leq p} n_i$ ($n_{\downarrow} = \min_{1 \leq i \leq p} n_i$).

Proof. According to Theorem 2, $\sigma(G) = (\bigcup_{i=1}^p \sigma(G_i) \setminus \{d_i\}) \cup \sigma(M)$. Then $\forall i \in \{1, \dots, p\}$ $\lambda_{n_i}(G_i) = d_i - s(G_i) \in \sigma(G)$ and hence

$$\lambda_n(G) \in \{d_i - s(G_i), i = 1, \dots, p\} \cup \{\lambda_n(M)\}.$$

Since $\lambda_1(G) = \lambda_1(M)$ (notice that $\lambda_1(G) \geq d_i \forall i \in \{1, \dots, p\}$), the equality (12) holds.

Now, we prove the inequality (13). Consider the symmetric matrix $M' = KA(H)K + D$ in (4), where $K = \text{diag}(\sqrt{n_1}, \dots, \sqrt{n_p})$, which is similar to the matrix M . Let (λ, x) be an eigenpair of H , where x is such that $\sum_{i=1}^p x_i^2 = 1$. Setting $y = K^{-1}x$, then

$$\begin{aligned} \lambda_n(G) &\leq \min \sigma(M) = \min \sigma(M') \\ &\leq \frac{y^T (KA(H)K + D) y}{y^T y} \\ &= \frac{x^T A(H)x + x^T K^{-1}DK^{-1}x}{x^T K^{-2}x} \\ &= \frac{\lambda x^T x + x^T DN^{-1}x}{\sum_{i=1}^p \frac{x_i^2}{n_i}} \\ &= \frac{\lambda + \sum_{i=1}^p \frac{d_i}{n_i} x_i^2}{\sum_{i=1}^p \frac{1}{n_i} x_i^2} \leq \lambda_1(M') \leq \lambda_1(G). \end{aligned}$$

Taking into account that $\tilde{d}_{\uparrow} = \max_{1 \leq i \leq p} \frac{d_i}{n_i}$ ($\tilde{d}_{\downarrow} = \min_{1 \leq i \leq p} \frac{d_i}{n_i}$) and $n_{\uparrow} = \max_{1 \leq i \leq p} n_i$ ($n_{\downarrow} = \min_{1 \leq i \leq p} n_i$), we may conclude the following.

- If $\lambda = \lambda_p(H)$, then $\lambda_n(G) \leq \frac{\lambda_p(H) + \tilde{d}_{\uparrow}}{\frac{1}{n_{\uparrow}}} = n_{\uparrow} (\lambda_p(H) + \tilde{d}_{\uparrow})$.
- If $\lambda = \lambda_1(H)$, then $\lambda_1(G) \geq \frac{\lambda_1(H) + \tilde{d}_{\downarrow}}{\frac{1}{n_{\downarrow}}} = n_{\downarrow} (\lambda_1(H) + \tilde{d}_{\downarrow})$.

Therefore, $s(G) \geq n_{\downarrow} (\lambda_1(H) + \tilde{d}_{\downarrow}) - n_{\uparrow} (\lambda_p(H) + \tilde{d}_{\uparrow}) = n_{\downarrow} (\lambda_1(H) + \tilde{d}_{\downarrow}) - n_{\downarrow} (\lambda_p(H) + \tilde{d}_{\uparrow}) - (n_{\uparrow} - n_{\downarrow}) (\lambda_p(H) + \tilde{d}_{\uparrow})$. ■

As immediate consequence of Theorem 7, we have the following corollary.

Corollary 5 *If the graph H has at least one edge and $G = \bigvee_H \mathcal{F}$, then*

$$s(G) \geq n_{\downarrow} \left(s(H) - (\tilde{d}_{\uparrow} - \tilde{d}_{\downarrow}) \right).$$

Proof. From (13), it follows

$$\begin{aligned} s(G) &\geq n_{\downarrow} \left(s(H) - (\tilde{d}_{\uparrow} - \tilde{d}_{\downarrow}) \right) - (n_{\uparrow} - n_{\downarrow}) \left(\lambda_p(H) + \tilde{d}_{\uparrow} \right) \\ &\geq n_{\downarrow} \left(s(H) - (\tilde{d}_{\uparrow} - \tilde{d}_{\downarrow}) \right) \end{aligned} \quad (14)$$

The inequality (14) is obtained taking into account that $\tilde{d}_{\uparrow} \leq 1$ and, since H has at least one edge, $\lambda_p(H) \leq -1$ and therefore, $(n_{\uparrow} - n_{\downarrow}) \left(\lambda_p(H) + \tilde{d}_{\uparrow} \right) \leq 0$. ■

Using this corollary, and taking into account that \tilde{d}_{\uparrow} and \tilde{d}_{\downarrow} are both in the interval $(0, 1)$, it follows that $s(G) \geq n_{\downarrow}(s(H) - 1)$.

Theorem 8 *If $G = \bigvee_H \mathcal{F}$, then*

$$s(G) \leq \max_{1 \leq i \leq p} d_i + \lambda_1(H) \lambda_1(P) - \min\{d_{i^*} - s(G_{i^*}), \lambda_p(M)\}.$$

Proof. By Theorem 2, $\sigma(G) = (\bigcup_{i=1}^p \sigma(G_i) \setminus \{d_i\}) \cup \sigma(M)$, where $M = D + A(H) \circ P$, with $D = \text{diag}(d_1, \dots, d_p)$, and \circ denotes the Hadamard product (see, for instance, [5]). Since when we have two symmetric nonnegative matrices of order p , A and B , $\lambda_1(A + B) \leq \lambda_1(A) + \lambda_1(B)$ and $\lambda_1(A \circ B) \leq \lambda_1(A \otimes B) = \lambda_1(A) \lambda_1(B)$, where \otimes is the Kronecker product, we may conclude that

$$\lambda_1(M) \leq \lambda_1(D) + \lambda_1(A(H) \circ P) \leq \lambda_1(D) + \lambda_1(H) \lambda_1(P) = \max_{1 \leq i \leq p} d_i + \lambda_1(H) \lambda_1(P).$$

Since $\lambda_n(G) = \min\{d_{i^*} - s(G_{i^*}), \lambda_p(M)\}$, it follows that,

$$s(G) \leq \max_{1 \leq i \leq p} d_i + \lambda_1(H) \lambda_1(P) - \min\{d_{i^*} - s(G_{i^*}), \lambda_p(M)\}.$$

■

Theorem 9 *If the graph H has at least one edge and $G = \bigvee_H \mathcal{F}$, then*

$$s(M) \leq s(G) < s(M) + \max_{1 \leq i \leq p} \{d_i\}.$$

Proof. By Theorem 7, $s(G) = s(M) + \max_{1 \leq i \leq p} \{\lambda_p(M) - \lambda_{n_i}(G_i), 0\}$.

1. If $\max_{1 \leq i \leq p} \{\lambda_p(M) - \lambda_{n_i}(G_i), 0\} = 0$, then the left inequality holds as equality and the right inequality is strict.
2. Otherwise, assume that $\exists i^* \in \{1, \dots, p\}$ such that $\max_{1 \leq i \leq p} \{\lambda_p(M) - \lambda_{n_i}(G_i), 0\} = \lambda_p(M) - \lambda_{n_{i^*}}(G_{i^*})$. Since,

$$\lambda_p(M) - \lambda_{n_{i^*}}(G_{i^*}) < -\lambda_{n_{i^*}}(G_{i^*}) \leq d_{i^*} \leq \max_{1 \leq i \leq p} \{d_i\},$$

then the right inequality holds. Notice that, when H has at least one edge,

$$\lambda_p(M) < 0. \text{ In fact, if } ij \in E(H), \text{ the matrix } B_{ij} = \begin{pmatrix} d_i & \sqrt{n_i n_j} \\ \sqrt{n_i n_j} & d_j \end{pmatrix}$$

is a principal submatrix of $P_{ij} M P_{ij}^T$, where P_{ij} is permutation matrix. Therefore, $\lambda_p(M) = \lambda_p(P_{ij} M P_{ij}^T) \leq \lambda_2(B_{ij}) < 0$. The left inequality follows from the fact that the eigenvalues of M are also eigenvalues of G . ■

3.4 An infinite family of non regular graphs of order n with spread equal to n .

Theorem 10 Consider the positive integres $p, q \geq 3$ and $n \in \mathbb{N}$ such that $n \geq p + q + 3$. Let $H = P_3$ and let $\mathcal{F} = \{G_1, G_2, G_3\}$ be a family of graphs, where $G_1 = C_p$, $G_2 = C_q$ and $G_3 = C_{n-p-q}$. If $\mathcal{S} = \{S_1, S_2, S_3\}$ is such that $S_i = V(G_i)$ for $i = 1, 2, 3$, then the graph

$$G = \bigvee_{(H, \mathcal{S})} \mathcal{F}. \quad (15)$$

is non regular and is such that $s(G) \leq n$. Furthermore, $s(G) = n$ if and only if $q = \frac{n}{2}$.

Proof. By definition of generalized join, it is immediate that G is non regular. By Theorem 2

$$\sigma(G) = \bigcup_{i=1}^3 (\sigma(G_i) \setminus \{2\}) \cup \{\beta_1, \beta_2, \beta_3\},$$

where β_i , with $i \in \{1, 2, 3\}$, are the roots of the characteristic polynomial of the matrix

$$M = \begin{pmatrix} 2 & q & 0 \\ p & 2 & n-p-q \\ 0 & q & 2 \end{pmatrix}.$$

Then $\beta_1 = 2, \beta_2 = 2 + \sqrt{q(n-q)}$, and $\beta_3 = 2 - \sqrt{q(n-q)}$. Notice that the largest eigenvalue of M is β_2 and $\lambda_{\min}(G) = \beta_3 = 2 - \sqrt{q(n-q)} < -2$ (taking into account the values of p, q and n and since $\lambda_{\min}(G_i) \geq -2$, for $i = 1, 2, 3$). Therefore,

$$s(G) = 2\sqrt{q(n-q)}.$$

Since $q(n-q) \leq \frac{n^2}{4}$ and $q(n-q) = \frac{n^2}{4}$ if and only if $q = \frac{n}{2}$, the result follows. ■

As immediate consequence of Theorem 10, if n is an even positive number not less than 12, $q = \frac{n}{2}$ and $3 \leq p \leq \frac{n-6}{2}$ then the graph G defined in (15) is such that $s(G) = n$.

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